Signals and systems

Fourier Transform Examples — lecture 19

① Find the Fourier transform of the following signals using convolution theorem.

Ⓐ $x(t) = e^{-2t} u(t) * e^{-5t} \cdot u(t)$

Ⓑ $x(t) = \frac{d}{dt}\left[e^{-2t} \cdot u(t) * e^{-5t} \cdot u(t)\right]$

Ⓒ $x(t) = \left[e^{-2t} u(t) * e^{-5t} \cdot u(t-5)\right]$

Determine $x(t)$ in all the above cases.

Ⓐ $x(t) = e^{-2t} \cdot u(t) * e^{-5t} u(t)$

$$X(j\omega) = F\left[e^{-2t} \cdot u(t)\right] \cdot F\left[e^{-5t} \cdot u(t)\right]$$

$$F\left[e^{-2t} u(t)\right] = \frac{1}{j\omega + 2} \quad ; \quad F\left[e^{-5t} \cdot u(t)\right] = \frac{1}{j\omega + 5}$$

$$X(j\omega) = \frac{1}{(j\omega+2)(j\omega+5)} = \frac{1}{3}\left[\frac{1}{j\omega+2} - \frac{1}{j\omega+5}\right]$$

$$x(t) = F^{-1}\left[X(j\omega)\right] = \frac{1}{3}\left[e^{-2t} u(t) - e^{-5t} u(t)\right]$$

$$x(t) = \frac{1}{3}\left[e^{-2t} - e^{-5t}\right] u(t)$$

Ⓑ $x(t) = \frac{d}{dt}\left[e^{-2t} \cdot u(t) * e^{-5t} u(t)\right]$

Let: $x_1(t) = e^{-2t} \cdot u(t) * e^{-5t} \cdot u(t)$

$$X_1(\omega) = \frac{1}{(j\omega+2)(j\omega+5)}$$

Using time differentration property of FT, we get

$$x(t) = \frac{d\, x_1(t)}{dt} \Rightarrow X(j\omega) = j\omega\, X_1(\omega)$$

$$X(j\omega) = \frac{j\omega}{(j\omega+2)(j\omega+5)}$$, using partial fraction expansion, we get

$$X(j\omega) = \frac{A_1}{j\omega+2} + \frac{A_2}{j\omega+5} \Rightarrow$$

$$j\omega = A_1(j\omega+5) + A_2(j\omega+2)$$

* Let $j\omega = -2$:

$A_1 = -\frac{2}{3}$

* let $j\omega = -5$

$A_2 = 5/3$

$$X(j\omega) = \frac{1}{3}\left[-\frac{2}{j\omega+2} + \frac{5}{j\omega+5}\right]$$

$$x(t) = F^{-1}\left[X(j\omega)\right] = \frac{1}{3}\left[-2 \cdot e^{-2t} + 5 \cdot e^{-5t}\right] u(t)$$

$$x(t) = \frac{1}{3}\left[-2e^{-2t} + 5 \cdot e^{-5t}\right] u(t)$$

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(c) $x(t) = e^{-2t} \cdot u(t) * e^{-5t} \cdot u(t-5)$

$$x(t) = x_1(t) * x_2(t)$$

$$X(j\omega) = X_1(\omega) \cdot X_2(j\omega)$$

$$X_1(j\omega) = \frac{1}{j\omega+2}$$

$$x_2(t) = e^{-5t} \cdot u(t-5) = e^{-25} \cdot e^{-5(t-5)} \cdot u(t-5)$$

$$X_2(\omega) = e^{-25} \cdot \frac{1}{j\omega+5}$$

$$X(j\omega) = e^{-25}\left[\frac{1}{(j\omega+2)(j\omega+5)}\right] = \frac{1}{3} \cdot e^{-25}\left[\frac{1}{j\omega+2} - \frac{1}{j\omega+5}\right]$$

$$x(t) = \frac{e^{-25}}{3}\left[e^{-2t} - e^{-5t}\right] \cdot u(t)$$

(2) Consider the following signals $x_1(t)$ and $x_2(t)$. Find

$$y(t) = x_1(t) * x_2(t)$$

$$x_1(t) = e^{-2t} \cdot u(t)$$

$$x_2(t) = e^{3t} \cdot u(-t)$$

$$X_1(\omega) = \frac{1}{j\omega+2} \qquad X_2(\omega) = -\frac{1}{j\omega-3}$$

$$x_1(t) * x_2(t) = X_1(\omega) \cdot X_2(\omega)$$

$$Y(j\omega) = \frac{1}{j\omega+2} \cdot \frac{-1}{j\omega-3} \Rightarrow Y(j\omega) = \frac{A_1}{j\omega+2} + \frac{A_2}{j\omega-3}$$

$$= \frac{1}{5}\left[\frac{1}{j\omega+2} - \frac{1}{j\omega-3}\right]$$

$$y(t) = F^{-1}\left[Y(j\omega)\right] = \frac{1}{5}\left[e^{-2t} \cdot u(t) + e^{3t} u(-t)\right]$$

$$y(t) = \frac{1}{5}\left[e^{-2t} u(t) + e^{3t} \cdot u(-t)\right]$$

(3) Consider the following differential equation. Determine the frequency response.

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t)$$

Taking FT on both sides of the above differential equation, we get :

$$(j\omega)^2 Y(j\omega) + 5(j\omega) \cdot Y(j\omega) + 6Y(j\omega) = [j\omega + 4] X(j\omega)$$

$$\frac{Y(j\omega)}{X(j\omega)} = H(j\omega) = \frac{j\omega+4}{(j\omega+2)(j\omega+3)}$$

(2)

4) A certain continuous linear time invariant system is described by the following differential equation.

$$\frac{dy(t)}{dt} + 5y(t) = x(t)$$

Determine $y(t)$, using FT for the following input signals

a) $x(t) = e^{-2t} \cdot u(t)$

b) $x(t) = 10\,u(t)$

c) $x(t) = \delta(t)$

a) $x(t) = e^{-2t} u(t)$

Taking FT on both sides we get

$$(j\omega + 5)\, Y(j\omega) = X(j\omega)$$

$$F[e^{-2t} \cdot u(t)] = \frac{1}{j\omega + 2}$$

$$Y(j\omega) = \frac{1}{(j\omega+2)(j\omega+5)} = \frac{1}{3}\left[\frac{1}{j\omega+2} - \frac{1}{j\omega+5}\right]$$

$$y(t) = F^{-1}[Y(j\omega)] = \frac{1}{3}\left[e^{-2t} - e^{-5t}\right] \cdot u(t)$$

$$y(t) = \frac{1}{3}\left[e^{-2t} - e^{-5t}\right] \cdot u(t)$$

b) $x(t) = 10\,u(t)$

$$X(j\omega) = F[10\,u(t)] = 10\pi\,\delta(\omega) + \frac{10}{j\omega}$$

$$Y(j\omega) = \frac{X(j\omega)}{j\omega + 5} = \left[\pi\,\delta(\omega) + \frac{1}{j\omega}\right] \cdot \frac{10}{j\omega+5}$$

$$= \frac{10\pi\,\delta(\omega)}{j\omega+5} + \frac{10}{j\omega(j\omega+5)}$$ → using partial fraction expansion.

$$\frac{10}{j\omega(j\omega+5)} = \frac{A_1}{j\omega} + \frac{A_2}{j\omega+5} \Rightarrow$$

$$A_1(j\omega+5) + A_2 \cdot j\omega = 10$$

* $j\omega = 0 \Rightarrow A_1 = 10/5 = 2$

* $j\omega = -5 \Rightarrow -5A_2 = 10 \Rightarrow A_2 = -2$

$$\frac{10}{j\omega(j\omega+5)} = \frac{2}{j\omega} - \frac{2}{j\omega+5}$$

$$Y(j\omega) = \frac{10\pi\,\delta(\omega)}{j\omega+5} + \frac{2}{j\omega} - \frac{2}{j\omega+5}$$

Applying the property $X(j\omega)\,\delta(\omega) = X(0)\cdot\delta(\omega)$ in the equation, we get :

$$Y(j\omega) = \frac{10}{5}\pi\delta(\omega) + \frac{2}{j\omega} - \frac{2}{j\omega+5}$$

$$= 2\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] - \frac{2}{j\omega+5}$$

$$y(t) = F^{-1}[Y(j\omega)] = 2\left[u(t) - e^{-5t}u(t)\right]$$

$$y(t) = 2\left[1 - e^{-5t}\right]u(t)$$

Note : $F^{-1}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] = u(t)$

* The above response is called "step Response" because the input $u(t)$ is a step signal.

c) $x(t) = \delta(t)$

Solution: $X(j\omega) = 1$

$$Y(j\omega) = \frac{1}{j\omega+5}$$

$$y(t) = F^{-1}[Y(j\omega)] = e^{-5t}\cdot u(t)$$

$$y(t) = e^{-5t}\cdot u(t)$$

* The above response is called "Impulse Response" of the system because the input $\delta(t)$ is an impulse.

5) Consider an LTI system with the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Assume Intial values are zeros.

Find the Frequency Response and impulse response.

Solution: Taking FT on both sides of the above equation, we get

$$(j\omega)^2Y(j\omega) + 4j\omega\,Y(j\omega) + 3Y(j\omega) = j\omega\,X(j\omega) + 2X(j\omega)$$

$$Y(j\omega)\left[(j\omega)^2 + 4j\omega + 3\right] = X(j\omega)\left[j\omega + 2\right]$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{(j\omega+2)}{[(j\omega)^2 + 4j\omega + 3]} = \frac{j\omega+2}{(j\omega+1)(j\omega+3)}$$

frequency Response.

To find the impulse response:

$x(t) = \delta(t)$

$$F[x(t)] = F[\delta(t)] = 1$$

$$Y(jw) = X(jw) \cdot H(jw) = 1 \cdot \frac{jw+2}{(jw+1)(jw+3)} = \frac{(jw+2)}{(jw+1)(jw+3)}$$

$$Y(jw) = \frac{jw+2}{(jw+1)(jw+3)} = \frac{A_1}{jw+1} + \frac{A_2}{jw+3}$$

$$jw+2 = A_1(jw+3) + A_2(jw+1)$$

let $jw = -1 \Rightarrow$

$$1 = 2A_1 \Rightarrow \boxed{A_1 = \frac{1}{2}}$$

let $jw = -3 \Rightarrow$

$$-1 = -2A_2 \quad \text{or} \quad \boxed{A_2 = \frac{1}{2}}$$

$$Y(jw) = \frac{1}{2}\left[\frac{1}{jw+1} + \frac{1}{jw+3}\right]$$

Taking inverse FT we get:

$$y(t) = F^{-1}[Y(jw)] = \frac{1}{2}\left[e^{-t} + e^{-3t}\right]u(t)$$

$$y(t) = \frac{1}{2}\left[e^{-t} + e^{-3t}\right]u(t)$$

6) Systems Described by differential equation:

Find the differentral equation for the system described by the following frequency response:

$$H(jw) = \frac{jw}{(1+jw)(2+jw)} = \frac{jw}{2+3(jw)+(jw)^2}$$

Solution: From the convolution property:

$$H(jw) = \frac{Y(jw)}{X(jw)} = \frac{jw}{2+3(jw)+(jw)^2}$$

Cross multiply:

$$[(jw)^2 + 3(jw) + 2]\, Y(jw) = jw\, X(jw)$$

We can inverse each term to obtain the time-domain relationship between $x(t)$ and $y(t)$

$$\underbrace{\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}}_{\text{differential equation of the system.}}$$